

# Interference with monochromatic light

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## 1 Two beam interference

### 1.1 Division of amplitude

For interference at least two beams are needed.

Two special situations of interference:

- two beam interference (i.e., two beams interfere),
- multiple beam interference (i.e., very many beams interfere).

There are two fundamental methods of generating interfering beams:

- division of amplitude,
- division of wavefront.

No chance to apply two or even many independent sources to get interference patterns which are stable in time; fluctuations of rates  $\gg 10^8$  hertz happen, observation impossible. Therefore, the interfering beams must have their origin in the same source. At first, the method of division of amplitude shall be applied to generate two interfering beams, Fig. 1.

Analyzing the interference pattern, different situations must be regarded:

- a) Parallel monochromatic light, all elements perfectly aligned and flat.  
Wave entering the beamsplitter  $BS$ :

$$E_{\text{in}} = E_0 e^{i(\omega t - kz_0)} \quad (1)$$

$E_{\text{in}}$  is divided into

$$\text{and} \quad \begin{aligned} E_1 &= |E_1| e^{i(\omega t - kz_1)} \text{ travelling } BS - M_1 - BS - \dots \\ E_2 &= |E_2| e^{i(\omega t - kz_2)} \text{ travelling } BS - M_2 - BS - \dots \end{aligned} \quad (2)$$

$E_2$  can also be thought travelling  $BS - M'_2 - BS - \dots$

$E_1$  and  $E_2$  superpose (interfere) from  $BS$  “downstream”. The resulting electric field oscillates too fast to be detected, only the intensity can be measured and yet only as a mean value averaged over the integration time

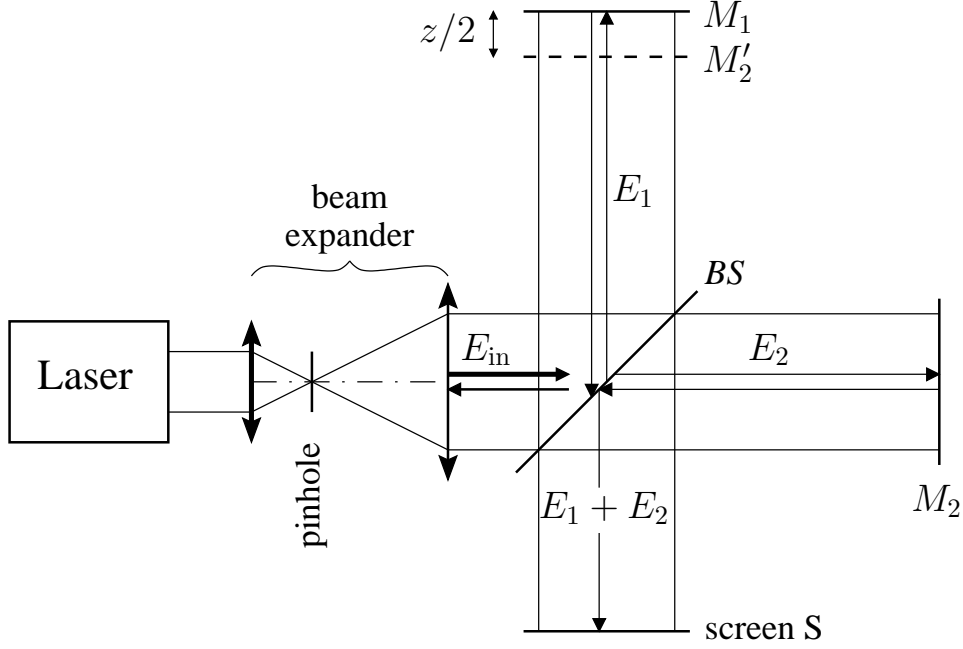


Figure 1: Generating two beams by division of amplitude in a Michelson interferometer

$BS$ : beamsplitter

$M_1, M_2$ : plane mirrors

$M_2'$ : virtual image of  $M_2$  generated by  $BS$

$BS$  does not modify the shape of the beam when dividing it (i.e., the wavefront keeps its form); only the amplitude is reduced by the dividing process.

of the detecting instrument (film, human eye, photo-multiplier, ...). So  $\langle |E_1 + E_2|^2 \rangle$  must be calculated. It is

$$|E_1 + E_2|^2 = (E_1 + E_2)(E_1^* + E_2^*) = |E_1|^2 + |E_2|^2 + E_1 E_2^* + E_1^* E_2 . \quad (3)$$

Introducing the intensity as

$$I = |E|^2$$

we get on the screen

$$\begin{aligned} |E_1 + E_2|^2 = I &= I_1 + I_2 + \sqrt{I_1 I_2} e^{i(-kz_1 + kz_2)} + \sqrt{I_1 I_2} e^{i(kz_1 - kz_2)} \\ &= I_1 + I_2 + 2\sqrt{I_1 I_2} \cos kz , \end{aligned} \quad (4)$$

where

$$2 \cdot \frac{z}{2} = z = z_2 - z_1$$

is the path difference of the two beams as can be seen on the sketch (Fig. 1). Obviously there is no need to calculate the temporal mean value of  $I$ , as

there is no dependence on  $t$ . This is because both  $\omega$ , i.e., in beams 1 and 2, are identical. Even if they are the same but not infinitely sharp as we supposed (monochromatic light), our result will contain  $t$  as a variable and integration over  $t$  will be necessary.

Equation (4), the interference signal, has a constant term and a cosine function, the latter is called interferogram. With

$$k = 2\pi\sigma n z , \quad (5)$$

$n$  being the refractive index on the path  $BS-M_1-BS$  resp.  $BS-M_2-BS$ , we state that at a given wave number  $\sigma$  and constant  $n$ , the brightness on the screen depends only on the path difference  $z$  of the beams. In particular, the screen is uniformly illuminated without any local structure, because  $z$  is constant all over the field of view. When mirror  $M_1$  or  $M_2$  is scanned along its normal the alignment precisely kept, the intensity observed on the screen follows the curve in Fig. 2. Introducing the visibility or contrast

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} ,$$

the interference signal can also be written

$$\frac{I}{I_1 + I_2} = 1 + \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} \cos 2\pi\sigma n z = 1 + V \cos 2\pi\sigma n z . \quad (6)$$

Thus the visibility is a measure of the difference of brightness between dark and bright fringes. To make it independent of the intensity of the entering beam it is divided by  $I_1 + I_2$  (normalization).

Notice that the displacement of  $M_1$  or  $M_2$ ,  $\Delta \frac{z}{2}$ , to proceed from a maximum to a minimum in the plot of Fig. 2 is tiny. Indeed, it is for red light at 632,8 nm (He-Ne laser) and  $n = 1$

$$\Delta \frac{z}{2} = \frac{1}{2} \cdot \frac{1}{2} \frac{\lambda}{n} = \frac{\lambda}{4} \approx 0,16 \mu\text{m} .$$

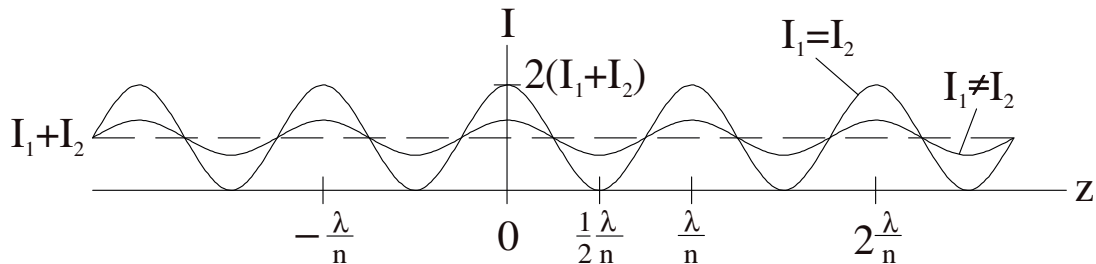


Figure 2: Intensity of interference as a function of path difference  $z$  at different ratios  $I_1/I_2$  but constant sum  $I_1 + I_2$ . The maximum of contrast is at  $I_1 = I_2$ .

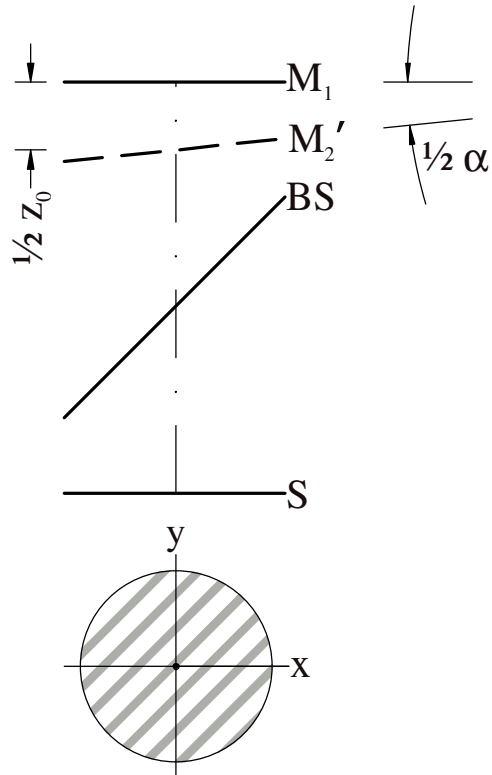


Figure 3: Simplified optical layout of the interferometer in Fig. 1 but  $M_2'$  inclined around  $x$  and  $y$  axis with respect to  $M_1$ . Fringes of equal thickness or Fizeau fringes as interference pattern.

- b) Parallel monochromatic light, all elements perfectly flat but  $M_1$  and  $M_2'$  mutually inclined

The analysis is like in paragraph a), however,  $z$  is now a function of  $x$  and  $y$ , the coordinates in the plane of the field of view. As the optical elements are flat,  $z$  is a linear function of  $x$  and  $y$ . It is obviously

$$z(x, y) = \alpha x + \beta y + z_0, \quad (7)$$

where  $\alpha$  and  $\beta$  are the angles at which the beams are mutually inclined. The path difference  $z$  is constant on straight lines parallel to the apex of the wedge which the plane wavefronts of beams 1 and 2 include. The thickness of the wedge is constant on these lines which is why they are called fringes of equal thickness; they are also named after Fizeau. The brightness across the fringe pattern is still the cosine function according to (4) and (6), resp. Wherever  $z$  is a multiple of  $\lambda/n$  there is the center of a bright fringe and this multiple is called interference order  $m$ . Every fringe has an individual  $m$ , and the equation defining a fringe is

$$\alpha x + \beta y + z_0 = m \frac{\lambda}{n}. \quad (8)$$

If mirror  $M_1$  or  $M_2$  is not perfectly plane,  $z$  has an additional dependence on  $x$  and  $y$  which is not necessarily linear. Lack of flatness combined with inclination leads to

$$\alpha x + \beta y + f(x, y) + z_0 = m \frac{\lambda}{n} \quad (9)$$

as definition of a fringe of interference order  $m$ , where the two-dimensional function  $f(x, y)$  is a measure of the contour of the mirrors  $M_1$  and  $M_2$ . Therefore Fizeau fringes are an excellent tool to control the flatness of optical elements in manufacturing workshops.

- c) Parallel monochromatic light, all elements perfectly flat,  $M_1$  and  $M_2'$  inclined or not, refractive index different along both paths and not constant. When the indices of refraction are constant but different on both paths, the exponentials in (4) must be replaced by

$$e^{i(-k_1 z_1 + k_2 z_2)} \quad \text{resp.} \quad e^{i(k_1 z_1 - k_2 z_2)},$$

with

$$k_1 z_1 = 2\pi\sigma n_1 z_1 \quad \text{and} \quad k_2 z_2 = 2\pi\sigma n_2 z_2 .$$

The product of the index of refraction and the geometrical path is called optical path. If  $n$  is not constant over the path length the product becomes an integral, the general form of which is

$$\int n(s) ds .$$

The path may be curved, this is why we have written  $s$ . Applied to the analysis of the interference pattern, Eq. (4) now becomes

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \left\{ 2\pi\sigma \left[ \int_{BS-M_2-BS} n_2(z_2) dz_2 - \int_{BS-M_1-BS} n_1(z_1) dz_1 \right] \right\} . \quad (10)$$

Beside  $z_1$  and  $z_2$ , which can depend on  $x$  and  $y$  as mentioned, even the functions  $n_1(z_1)$  and  $n_2(z_2)$  may change with  $x$  and  $y$ . As an example imagine a jet of gas traversing one of the interferometer arms.

- d) Monochromatic light, this time divergent or convergent, all elements flat and aligned, constant and equal indices of refraction. The bold path difference in Fig. 4, the bishop's mitre, is

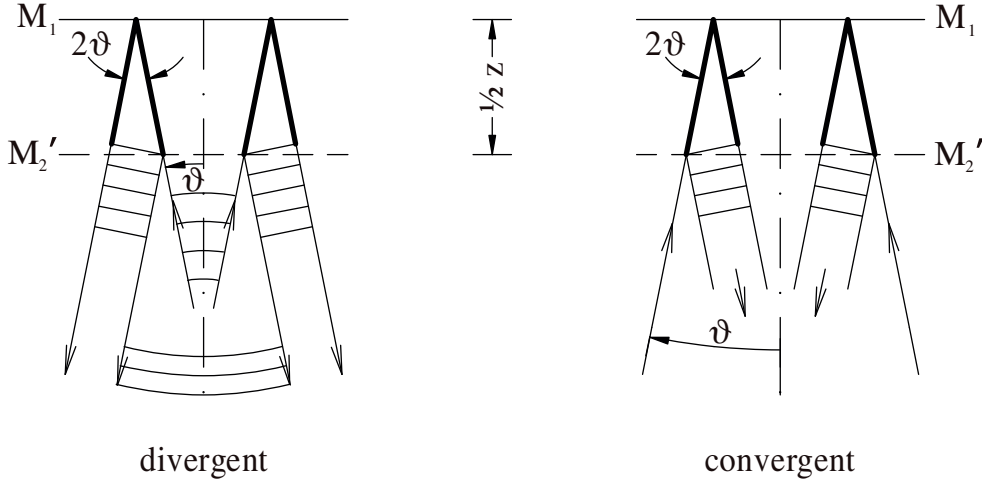


Figure 4: For divergent as well as convergent light the path difference is the bold figure, which looks like a bishop's mitre

$$\begin{aligned}
 & \frac{\frac{1}{2}z}{\cos \vartheta} + \frac{\frac{1}{2}z}{\cos \vartheta} \cos 2\vartheta \\
 &= \frac{z}{2 \cos \vartheta} (1 + \cos^2 \vartheta - \sin^2 \vartheta) \quad \left( \begin{array}{l} \text{because of} \\ \cos 2\vartheta = \cos^2 \vartheta - \sin^2 \vartheta \end{array} \right) \\
 &= \frac{z}{2 \cos \vartheta} 2 \cos^2 \vartheta \\
 &= z \cos \vartheta \quad \text{with } 0 < \vartheta < \vartheta_{\max} .
 \end{aligned} \tag{11}$$

Now the interference signal is

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(2\pi\sigma n z \cos \vartheta) . \tag{12}$$

The pattern consists of concentric circles  $\vartheta = \text{const.}$ , named fringes of equal inclination or Haidinger rings.

## 1.2 Division of wavefront

Field at  $P$

from slit 1:  $E_1 = |E_1| e^{i(\omega t - kr_1)}$

from slit 2:  $E_2 = |E_2| e^{i(\omega t - kr_2)}$

Intensity of interference field at  $P$ :

$$\begin{aligned}
 I &= |E_1 + E_2|^2 = |E_1|^2 + |E_2|^2 + E_1 E_2^* + E_1^* E_2 , \quad \text{with } |E_{1,2}|^2 = I_{1,2} \\
 \text{or} \\
 I &= I_1 + I_2 + 2\sqrt{I_1 I_2} \cos 2\pi\sigma n(r_2 - r_1) .
 \end{aligned} \tag{13}$$

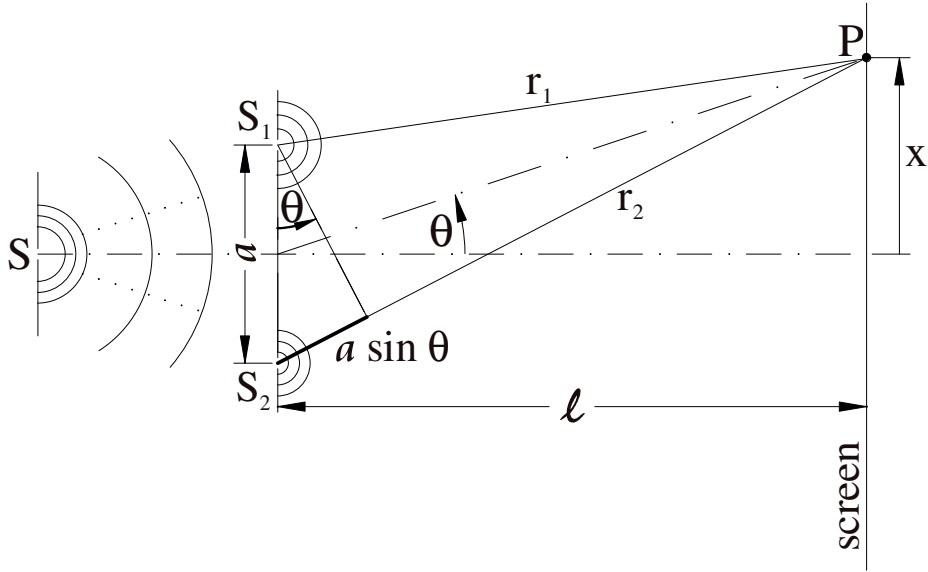


Figure 5: Young's interference experiment, the double slit. Monochromatic waves from two slits,  $S_1$  and  $S_2$ , interfere.  $S_1$  and  $S_2$  are infinitely narrow, and so is the source  $S$ . The observer,  $P$ , is very far away, i.e.,  $\ell \gg a$  (far-field approximation).

This is the same result as before at division of amplitude. If  $S$ ,  $S_1$ , and  $S_2$  are slits perpendicular to the plane of the sketch in Fig. 5, the interference pattern on the screen consists of straight cosine fringes parallel to  $S$ ,  $S_1$ , and  $S_2$ . When the screen of observation is far away so that  $\ell \gg a$  and if, in addition, the observer's position is restricted to  $\sin \theta \approx \tan \theta \approx \theta \approx \frac{x}{\ell}$ , then we get from (13) and  $r_2 - r_1 \approx a \sin \theta$ :

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\left(2\pi\sigma n \frac{a}{\ell} x\right). \quad (14)$$

The spacing of neighbour fringes is

$$\Delta x = \frac{\lambda \ell}{na}.$$

As a result of this analysis of two beam interference we conclude:

- Either method to generate interference (division of amplitude and division of wavefront) leads to cosine fringes. The geometrical form of the fringes is defined by the interference device (flatness of elements etc.) and by the divergence of the beam on the input side.
- Till now we have been fixed on two unrealistic restrictions, i.e., monochromatic light from point-like sources. As for the latter, parallel light is exclusively obtained from a point source at infinity or from such a source at the focal point of a perfect lens. And yet, even our admission of divergent

and convergent light in paragraph 1.1d required a point-like source resp. a point-like image (case of convergent light), for the interference phase

$$\Phi(x, y) = 2\pi\sigma n z \cos\vartheta(x, y)$$

in the interferogram term (12) is an unambiguous function of a point  $x, y$  on the observation screen only if the source consists of a point. As soon as the source is laterally extended, the same point  $x, y$  on the screen is attained by many light rays, all with different  $\vartheta$ . It is obvious that the contrast in the interference pattern then must decrease. Loss of contrast will also be stated, if the light is no longer monochromatic, because again the phase  $\Phi(x, y)$  at the point of observation,  $x, y$ , is not unambiguous if the wave number  $\sigma$  has more than a single value.

In the chapter on coherence we must bring the pattern back to reality based on an extended non-monochromatic source.

## 2 Multiple beam interference

Again the two methods of creating beams that can interfere are applied, i.e., division of amplitude and division of wavefront.

Example for division of amplitude: the Fabry–Pérot interferometer, example for division of wavefront: the diffraction grating.

Our analysis shall be restricted to the Fabry–Pérot interferometer, because the grating can be better treated with the concept of diffraction and Fourier optics. The Fabry–Pérot is nearly infinitely often applied, it is the resonator in a laser.

The basic realization of the Fabry–Pérot interferometer has plane mirrors as shown in Fig. 6. The mirrors are parallel, their coatings have a reflection coefficient  $R$  and a transmission ratio  $T$ , both related to the intensity,  $R$  is close to unity.

A parallel beam of monochromatic light enters from the left side under the angle of incidence  $\vartheta$ . Due to multiple reflection between the mirrors there is an infinite number of beams leaving the device to the right side; they are all laterally displaced by the same amount with respect to their neighbour. We suppose a small angle  $\vartheta$ , so that the displacement is negligible. There is a phase shift  $\alpha_T$  caused by the bare passage through the coating on the mirror in the direction glass–coating–medium between the mirrors. At each reflection on the coating surface a phase shift  $-\alpha_R$  is given to the wave, when it enters the coating from the side of that medium. Two successive beams leaving the interferometer to the right side differ by the bold geometrical path  $\delta = a \cos\vartheta$  (Fig. 6), which we know from Fig. 4, paragr. 1.1d). It is again bishop’s mitre, this time in horizontal position (sleeping bishop).



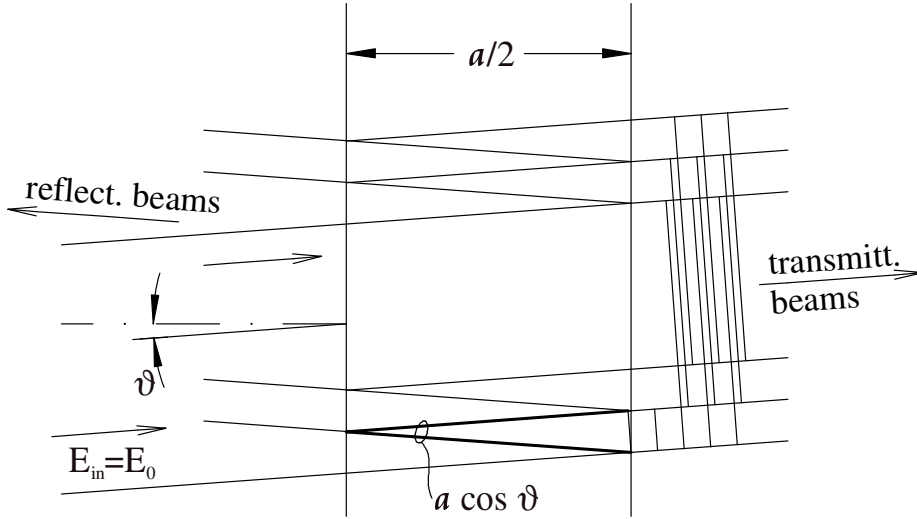


Figure 6: Multiple beam interference with parallel light. In transmission as well as in reflection there is an infinite number of interfering beams. The phase shift between two successive beams is always the same and corresponds to the bold path

The total transmitted light is the sum of the following beams with these complex amplitudes:

$$\text{1st beam: } E_1 = E_0 \sqrt{T} e^{i\alpha T} e^{i\frac{a}{2\cos\vartheta}} \sqrt{T} e^{-i\alpha T} = e^{i\frac{a}{2\cos\vartheta}} T E_0$$

$$\text{2nd beam: } E_2 = E_1 \sqrt{R} e^{-i\alpha_R} e^{-i2\pi\sigma n\delta} \sqrt{R} e^{-i\alpha_R} = \left[ R e^{-i(2\alpha_R + 2\pi\sigma n\delta)} \right] E_1$$

$$\text{3rd beam: } E_3 = \left[ R e^{-i(2\alpha_R + 2\pi\sigma n\delta)} \right] E_2 = \left[ R e^{-i(2\alpha_R + 2\pi\sigma n\delta)} \right]^2 E_1$$

⋮  
⋮  
⋮

The sum of these amplitudes is the geometrical series

$$\begin{aligned} \frac{E}{E_0} &= e^{i\frac{a}{2\cos\vartheta}} T \left( 1 + R e^{-i(2\alpha_R + 2\pi\sigma n\delta)} + \left[ R e^{-i(2\alpha_R + 2\pi\sigma n\delta)} \right]^2 + \dots \right) \\ &= e^{i\frac{a}{2\cos\vartheta}} \frac{T}{1 - R e^{-i\phi}}, \quad \text{with } \phi = 2\alpha_R + 2\pi\sigma n\delta. \end{aligned} \quad (15)$$

To obtain this result we have taken into account that the series has infinitely many terms, so it is

$$\left[ R e^{-i(2\alpha_R + 2\pi\sigma n\delta)} \right]^m \rightarrow 0$$

for  $m \rightarrow \infty$  and  $R < 1$ . The intensity is

$$\begin{aligned} \frac{I}{I_0} &= \frac{E \cdot E^*}{E_0^2} = \frac{|E|^2}{E_0^2} = \frac{T^2}{(1 - R e^{-i\phi})(1 - R e^{i\phi})} = \dots \\ &= \frac{T^2}{(1 - R)^2} \frac{1}{1 + M \sin^2 \frac{\phi}{2}}, \quad \text{with } M = \frac{4R}{(1 - R)^2}. \end{aligned} \quad (16)$$

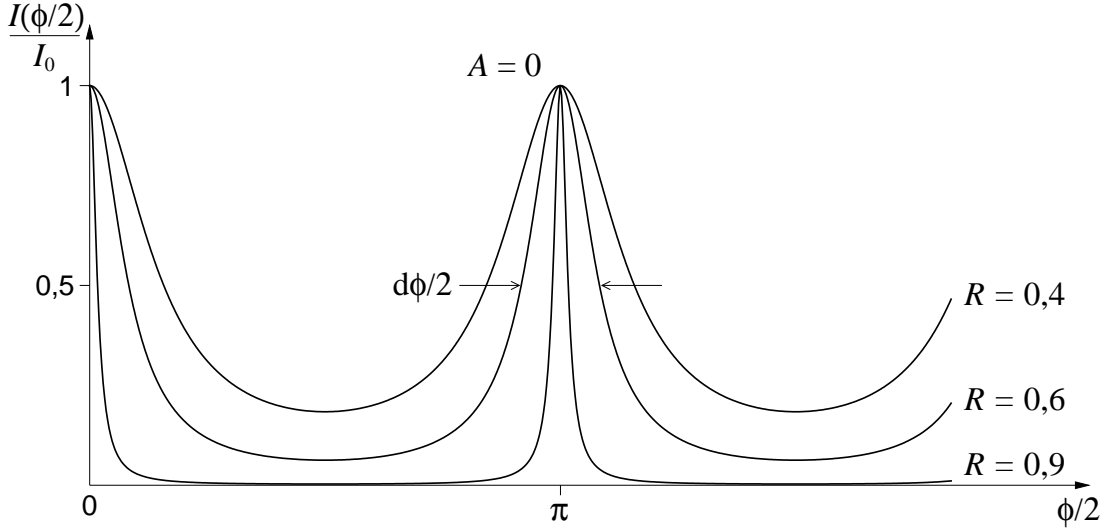


Figure 7: The Airy function for different reflection coefficients. The plot is the same, when  $\sigma$ ,  $n$ ,  $a$ , or  $\cos \vartheta$  is the abscissa because of  $\phi = 2\pi\sigma na \cos \vartheta$  ( $\alpha_R$  set to zero). Without absorption in the mirror coatings it is  $T = 1 - R$ , so the peaks on the plot attain 1

Equation (16) is called Airy function, it is plotted in Fig. 7 for different reflection coefficients  $R$ .

$\phi$  is a linear function of  $\sigma$ ,  $n$ ,  $a$ ,  $\cos \vartheta$ , actually

$$\phi = 2\alpha_R + 2\pi\sigma na \cos \vartheta ,$$

where  $\alpha_R$  can be set equal to zero, it is just a constant shift of the plot. The plot is the same for the four variables. The closer the reflection coefficient  $R$  is to unity the sharper are the maxima of the Airy function.

In the peaks where the Airy function equals one, there is a standing wave in the medium between the mirrors. Indeed, it follows from the relation above for the peaks

$$\frac{\phi}{2} = m\pi = \pi\sigma na \cos \vartheta$$

or

$$n \frac{a}{2} \cos \vartheta = m \frac{\lambda}{2} , \quad (17)$$

which means that  $\lambda/2$  fits  $m$  times ( $m$  integer) in the “optical gap”  $n \frac{a}{2} \cos \vartheta$  between the mirrors,  $\vartheta$  is supposed to be very small. Relation (17) indicates resonance, and this is why the Fabry–Pérot interferometer is also called resonator or cavity.

The factor  $\left(\frac{T}{1-R}\right)^2$  in (16) can be replaced by  $\left(\frac{T}{T+A}\right)^2$ , because of the relation

$$R + T + A = 1 ,$$

where  $A$  is the absorption of the coating on the mirrors. For material without absorption the factor becomes unity. Such materials are dielectric, and a whole stack of different types with individual thickness is needed to get a coating with the required reflection coefficient. However, the reflection of such a multifold changes rapidly with wavelength. Metallic single films absorb a little, but their spectrum of transmission is broad. In visible light silver is chosen, in ultraviolet aluminum is better.

The extrema of the Airy function are obviously

$$\begin{aligned} \frac{I_{\max}}{I_0} &= \left( \frac{T}{1-R} \right)^2 = \left( \frac{1-R-A}{1-R} \right)^2 \longrightarrow 1 \text{ for } A=0 \\ \frac{I_{\min}}{I_0} &= \left( \frac{T}{1-R} \right)^2 \frac{1}{1 + \left( \frac{4R}{1-R} \right)^2} = \left( \frac{T}{1+R} \right)^2 = \left( \frac{1-R-A}{1+R} \right)^2 \longrightarrow \left( \frac{1-R}{2} \right)^2 \\ &\text{for } A=0, R \rightarrow 1. \end{aligned} \quad (18)$$

Divergent light after passing a pair of Fabry–Pérot mirrors presents beautiful concentric circles as interference pattern. They are extremely sharp if  $R$  is close to unity. As  $\vartheta$  is constant on such a ring, they are again named fringes of equal inclination or Haidinger rings. Like in two beam interference, fringes of equal thickness can be generated, too. One of the mirrors must be slightly inclined and the beam has to be parallel.

From the sharpness of interference fringes one can immediately see whether they are generated by two beam or multiple beam interference. Two beam interference yields cosine fringes, whereas from multiple beam interference extremely sharp patterns are obtained, they look as if they were grooved with a needle.

In order to have a measure of the sharpness the finesse has been introduced, see Fig. 8; it is the ratio of the distance between two neighbour peaks and the halfwidth of a single one, i.e.,

$$F = \frac{\Delta \frac{\phi}{2}}{d \frac{\phi}{2}}.$$

The halfwidth of a peak is immediately obtained from

$$\frac{1}{2} = \frac{1}{1 + M \sin^2 \left( \frac{1}{2} d \frac{\phi}{2} \right)} \quad \text{with } M = \frac{4R}{(1-R)^2}$$

as

$$d \frac{\phi}{2} = \frac{2}{\sqrt{M}} = \frac{1-R}{\sqrt{R}} = 2\pi n a \cos \vartheta d\sigma, \quad (19)$$

where the last part of this equation has been taken from the definition of  $\phi/2$ . As  $\Delta \phi/2$  equals  $\pi$  (Fig. 8), the finesse results to

$$F = \frac{\pi \sqrt{R}}{1-R}. \quad (20)$$

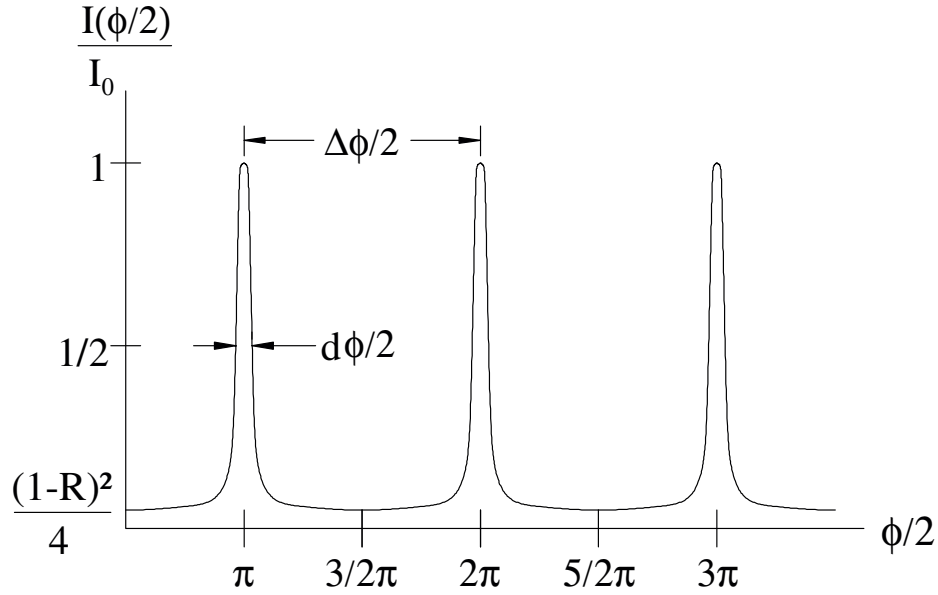


Figure 8: Halfwidth and free spectral range

The finesse can be pushed very high ( $F > 200$ ). But to achieve such a value the mirrors must be extremely plane, about  $\lambda/1000$ , and their alignment must be of the same precision.

The free spectral range,  $\Delta\sigma = \sigma_{\max} - \sigma_{\min}$ , fitting into the gap between two successive maxima of the Airy function (Fig. 8) is found from

$$\Delta\frac{\phi}{2} = 2\pi na \cos\vartheta \Delta\sigma = \pi \quad \text{as} \quad \Delta\sigma = \frac{1}{2na \cos\vartheta} .$$

As mentioned, the Fabry–Pérot serves as a laser cavity. The stimulated emission is bound to such frequencies which correspond to the peaks of the Airy function. Therefore a laser can simultaneously oscillate at those wavelengths where the resonance condition is fulfilled, that means

$$m \frac{\lambda}{2} = nL ,$$

where  $L$  is the length of the cavity. The corresponding frequencies are

$$\nu = m \frac{c}{2nL} ,$$

and

$$\Delta\nu = \frac{c}{2nL}$$

is the separation of longitudinal modes.