Quantum process tomography from incomplete data

Mário Ziman

Research Center for Quantum Information, Institute of Physics SAS, Bratislava

http://www.quniverse.sk/

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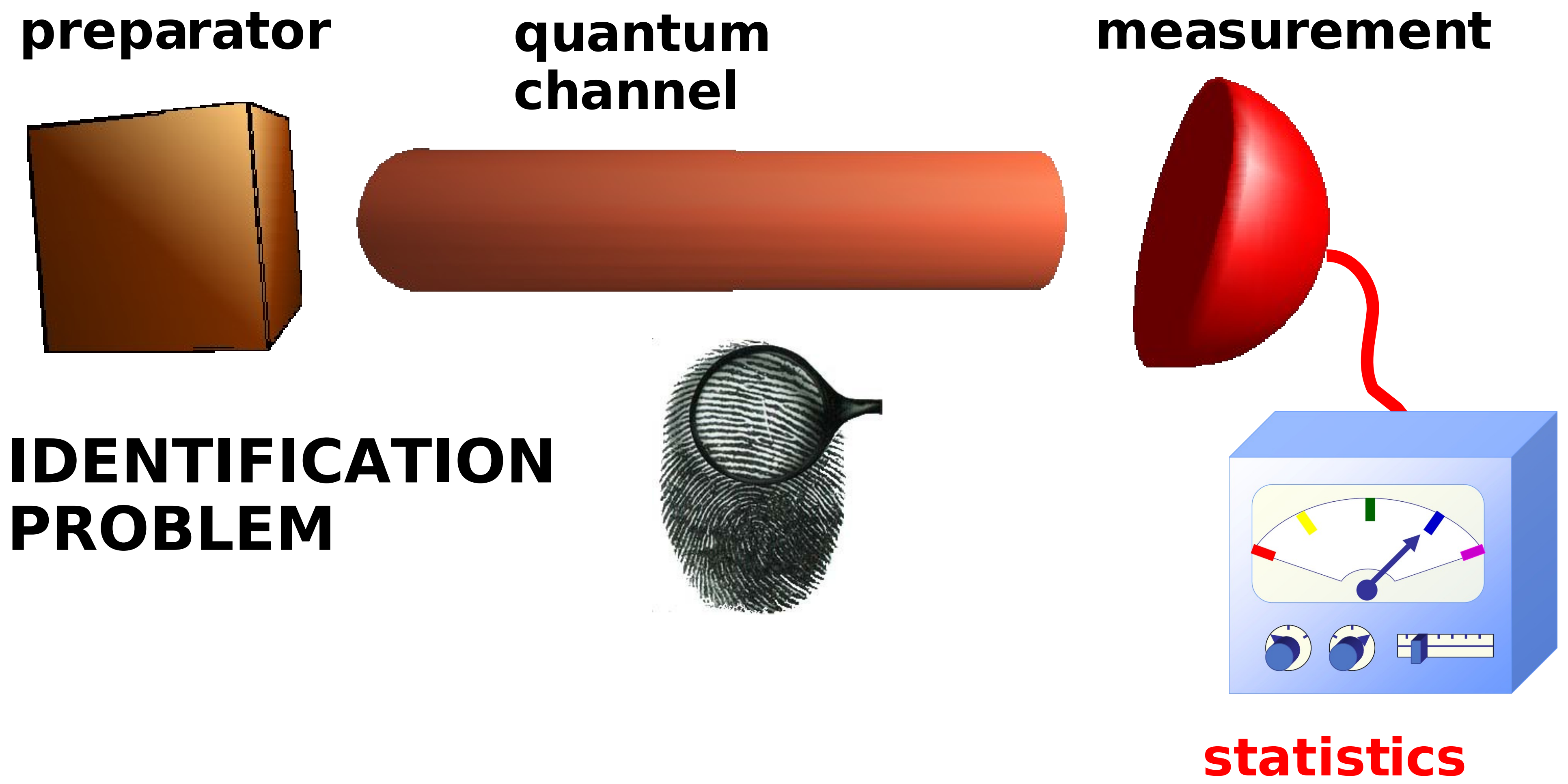
- process tomography
- “unphysical” results and maximum likelihood
- possible sources of “unphysicality” (speculations)
- incomplete data and MaxEnt principle
- MaxEnt for process tomography

Simplified experimental setup



Process identification – inverse problem

- **quantum theory** $\mathcal{Q}\left[(\mathcal{E},\rho_j,F_k)\right]=p_{jk}$
- **preparator identification** ρ
- **application of the channel** $\rho'=\mathcal{E}\otimes\mathcal{F}\left[\rho\right]$
- **measurement POVM** $\{F_k\}$ $\quad p_k(\rho)=\mathrm{Tr}\rho' F_k$
- **processing of exp. data to learn** $\mathcal{E}$

  - inverse process reconstruction $\mathcal{R}_{\mathrm{inv}}\left[(\rho_j,F_k,p_{jk}\right]=\mathcal{E}$

  - statistical process estimation $\mathcal{R}\left[(\rho_j,F_k,\mathrm{data})\right]=\mathcal{E}$

Process identification difficulties

- **inverse process reconstruction** $\mathcal{R}_{\text{inv}}[(\rho_j, F_k, p_{jk}] = \mathcal{E}$

  - system of linear equations

  - easy to implement

  - "unphysical" results

- **statistical process estimation** $\mathcal{R}\,[(\rho_j, F_k, \text{data})] = \mathcal{E}$

  - maximum likelihood methods

  - physical result is guaranteed

  - difficult optimization problem (exp)

Unphysical results

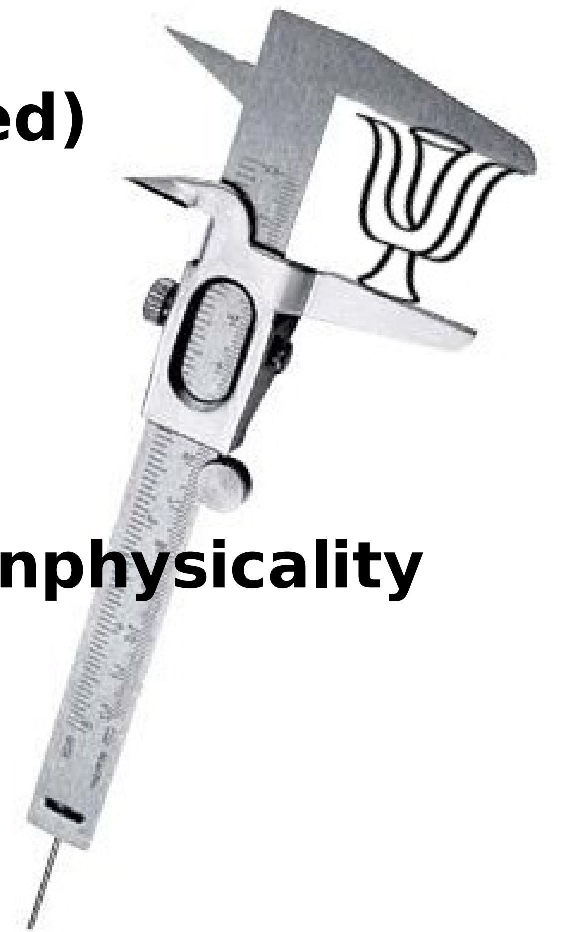
- problems with probabilities (small statistics)
- problems with experimental setup

  - preparator (tomography, uncorrelated)

  - measurement device (calibration)

  - noise can be included in description

  - memory effects
- imperfect preparators vs “artificial” unphysicality



Model of open system dynamics

- completely positive tracepreserving linear maps
  - addition of fixed uncorrelated ancilla
  - unitary system dynamics of system+ancilla
  - discarding the ancilla
- composition of three maps $\mathcal{E} = \mathcal{T}_{\mathrm{anc}} \circ \mathcal{U} \circ \mathcal{P}$
- preparation map $\mathcal{P} : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}_{\mathrm{anc}}$
  - potential source of “unphysicality”
  - CP maps $\Leftrightarrow$ $\mathcal{P}[\rho] = \rho \otimes \xi_{\mathrm{fixed}}$
  - ?verification? of preparators

Accessible transformations

- **all those for which the ancilliary model exists**
- **characterization:**

  - arbitrary mapping, $f : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$**, i.e.**

$$\rho \rightarrow \rho' = f(\rho)$$

  - implementation $\mathcal{P}[\rho] = \rho \otimes f(\rho)$

  - SWAP gate $\rho' = \mathrm{Tr}_{\mathrm{anc}}[U_{\mathrm{SWAP}} \rho \otimes f(\rho) U_{\mathrm{SWAP}}] = f(\rho)$

- **very artificial construction**
- **is it a bad news?**
- **linear accessible transformations**

Universal NOT in a lab

- **experimental situation:**
  - **- black box and qubits (let's say spins)**
  - **- preparation: SG measurement**
  - **- observation: outputs orthogonal to inputs**
- **is it unphysical?**
  - **- given qubits are entangled to qubits in the black box (singlet)**
  - **- interaction via SWAP gate**

Preparator devices

- independence of preparators and channels
- insight into physics behind the preparation
- E1:

  - preparator of ground state

  - all pure states prepared via unitary processing
- E2: intermediate dynamical map $\mathcal{E}_{t_1,t_2} = \mathcal{E}_{t_2} \circ \mathcal{E}_{t_1}^{-1}$

  - linear trace and hermiticity preserving + ???

  - NOT cannot be realized within this model

Complete process tomography

- **for experiments only the linearity is important**
- **number of parameters =** $d^2(d^2-1)$

  - exponential in number of qubits
- **based on (incomplete) state tomography**
- **test states = lin. independent states** $\{\rho_1,\ldots,\rho_{d^2}\}$

  - ***d*****x*****d*** **state reconstructions** $\rho_j \rightarrow \rho'_j = \mathcal{E}[\rho_j]$
- **ancilla-assisted tomography**

  - ancilla reduces the number of test states

  - single test state $\Omega_{\mathcal{E}} = \mathcal{E}\otimes I[\Psi_{ME}] = \frac{1}{d}\sum_{jk}\mathcal{E}[e_{jk}]\otimes e_{jk}$

Ancilla-assisted tomography

- **ancilla-assisted test state**

$$\Omega_{\mathcal{E}} = \mathcal{E} \otimes I[\Psi_{ME}] = \frac{1}{d}\sum_{jk} \mathcal{E}[e_{jk}] \otimes e_{jk} \qquad e_{jk} = |j\rangle\langle k|$$

- **state reconstruction of** $\Omega_{\mathcal{E}}$ **→ process**

$$\mathcal{E}[\rho] = \mathrm{Tr}_2[(I \otimes \rho^T)\Omega_{\mathcal{E}}]$$

- **faithful (admissible) states** $\Omega = \sum \omega_{\mu\nu} S_\mu \otimes S_\nu$

$$\Omega_{\mathcal{E}} = \sum \omega_{\mu\nu} \mathcal{E}[S_\mu] \otimes S_\nu = \sum \omega'_{\mu\nu} S_\mu \otimes S_\nu$$

$$\{S_0, \ldots, S_{d^2-1}\}$$

$$\mathcal{E}[S_\mu] = \sum \mathcal{E}_{\mu\nu} S_\nu$$

$$\mathcal{E}_{\mu\nu} = \mathrm{Tr} S_\mu^\dagger \mathcal{E}[S_\nu]$$

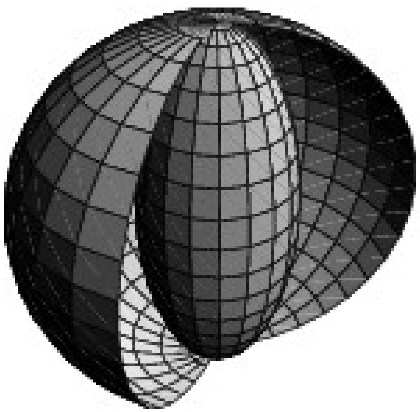
- **process tomography**

$$[\omega'] = [\mathcal{E}].[\omega] \;\Rightarrow\; [\mathcal{E}] = [\omega'].[\omega]^{-1}$$

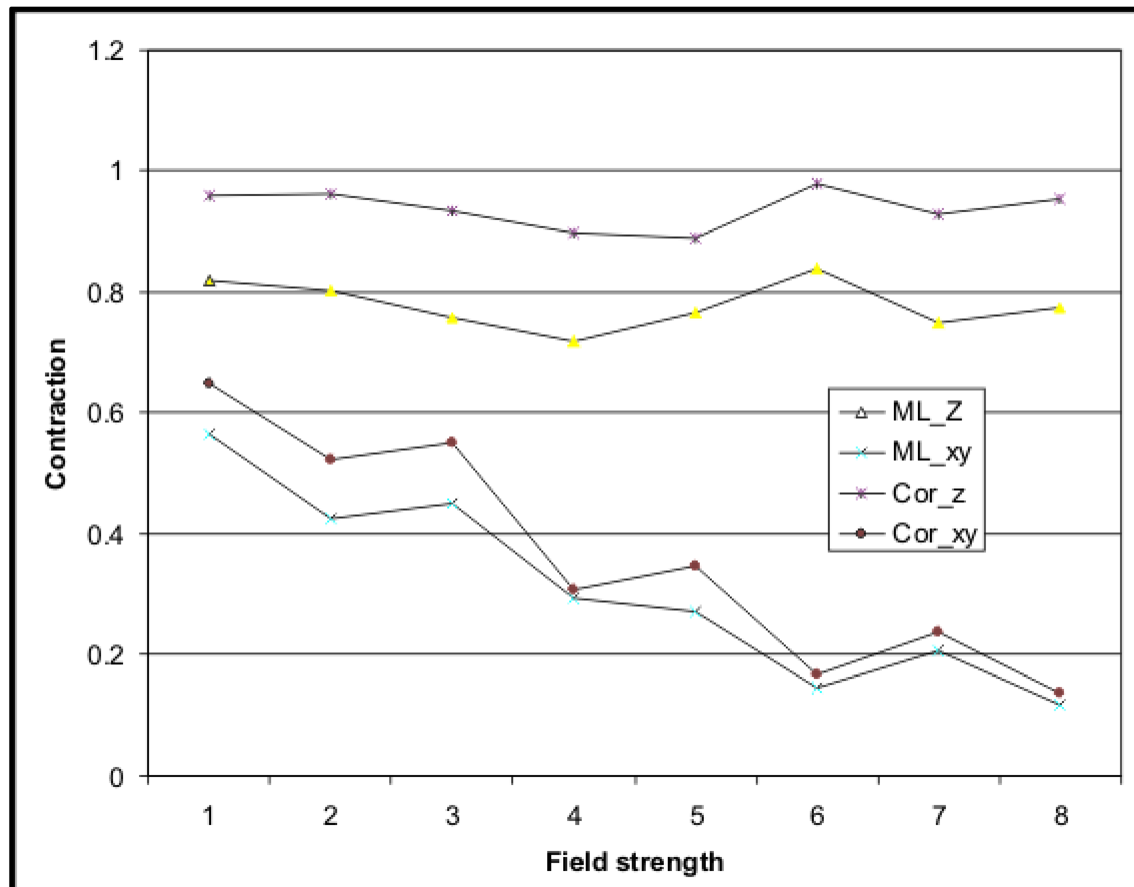
1. D'Arianno (Phys.Rev.A, 2001), Raginsky (Phys.Rev.A, 2001)

Qubit process tomography

- **phase damping** $\mathcal{E}^{(\lambda)}_{ph.damp} : \vec{r} \rightarrow \vec{r}\,' = (\lambda x, \lambda y, z)$



- **qubit=ion $_{171}Yb^{+}$ (Ch.Wunderlich)**



Incomplete knowledge on outcome states

- **exp.data do not determine all parameters**
- **observation level** $O = \{\Lambda_1, \ldots, \Lambda_K\}$ **(**$K < d^2 - 1$**)**

  - probabilities/mean values $r_k = \mathrm{Tr}\rho\Lambda_k = \langle\Lambda_k\rangle_\rho$
- **unknown mean values**
- **what is the state?** $\rho = \frac{1}{d}(I + \vec{r}\cdot\vec{\Lambda})$

  - no unique answer

  - needs some extra assumption/postulate
- **zero observation level**

  - each state equally probable ⇨ av. state ⇨ $\overline{\rho} = \frac{1}{d}I$
- **naïve strategy … set unknown parameters to 0**

Maximum entropy principle

- **E.T.Jaynes**

Instead of asking *what is the state* we should ask *what state <u>best</u> describes the state of <u>our knowledge</u> about the physical situation.*

- **average ⇔ entropy** $S(\rho) = -\mathrm{Tr}\rho\log\rho$
- **MaxEnt = choose the state with maximal entropy**

given the observation level constraints

$$\rho = \arg\max_{\rho}\{S(\rho) | \langle\Lambda_j\rangle_\rho = r_j, \forall j = 1,\ldots,K\}$$

Incomplete PT: 0 knowledge

- **What is the channel?**
- **average channel** $\mathcal{A}$

  - problem with measure

  - unital, because $\mathcal{E}_{\pm}: \vec{r} \rightarrow \vec{r}' = T\vec{r} \pm \vec{t}$ **are CP maps**

  - ***U*** **symmetry ⇨** $\mathcal{A}_{\mu} = \mu I + (1-\mu)\mathcal{A}$ **with** $\mathcal{A}[\rho] = \frac{1}{d} I \quad \forall \rho$

  - contraction to total mixture
- **no concept of channel entropy**

  - capacity, minimal output entropy, distance,

  - Jamiolkowski isomorphism (ancilla-assisted PT)

Incomplete PT: naïve approach

- transform states into total mixture
- analysis done for single qubit channels

  - no problem for single test state (no ancilla)

  - two/three test states numerically
- MaxEnt for states cannot be used directly
- problem: incompatible state transformations
- state MaxEnt for ancilla-assisted tomography

Incomplete PT: MaxEnt

- concept of state entropy for ancilla-assisted PT
- extension to nonancilliary approach

$$\boxed{(\rho_k, A) \leftrightarrow A \otimes X_k}$$

$$\langle A \rangle_{\mathcal{E}[\rho_k]} = \langle A \otimes X_k \rangle_{\mathcal{E}\otimes I[\Omega]}$$

$$\langle \mathcal{E}^*[A] \rangle_{\rho_k} = \langle \mathcal{E}^*[A] \otimes X_k \rangle_{\Omega}$$

$$\sum (\rho_k)_{ab} (\mathcal{E}^*[A])_{ba} = \sum \omega_{ab,cd} (X_k)_{dc} (\mathcal{E}^*[A])_{ba}$$

$$\Downarrow$$

$$\boxed{\vec{X}_k = [\Omega]^{-1} \vec{\rho}_k}$$

- for max. entangled state $X_k = \frac{1}{d} \rho_k^T$
- problem which Ω to use

Incomplete PT: qubit channel MaxEnt

- **0 knowledge ... contraction to total mixture**
- **single measurement, i.e.** $\mathcal{O}_{\mathrm{proc}} = \{(\varrho, F)\}$

  - data $\varrho = \frac{1}{2}I,\ F = \vec{f}\cdot\vec{\sigma}, m = \mathsf{Tr} F\varrho'$

  - estimated channel $\mathcal{E}_{\mathrm{est}}[\varrho] = \frac{1}{2}(I + m\vec{f}\cdot\vec{\sigma})$
- **analytically difficult**

Hypothesis testing

- **problem: find unique property (a priori info)**
- **small number of measurements**
- **quantify validity of the hypothesis**
- **H1: pure state preparator ψ**

  - test = single projective measurement
- **H2: unitary transformation *U***

  - AAPT with single projective measurement
- **H3: extremal channels**
- **H4: entanglement?**

Conclusion

- complete tomography is expensive
- incomplete MaxEnt is questionable
- hypothesis testing
  - pure state verification
  - contraction to pure state
  - testing for unitaries
- “golden standards” for state and process est.
- standards for calibration

Literature

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